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Nonlinear Passive Evolution Equations

D. WEXLER

Department of Mathematics,
Facultés Universitaires N.D. de la Paix, 61, Rue de Bruxelles, Namur, Belgium

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1. INTRODUCTION

Throughout this paper A is a maximal monotone (possibly nonlinear, unbounded, and multivalued) operator of a real Hilbert space H with domain $D(A)$ and range $R(A)$. The inner product in H will be denoted by $(\cdot, \cdot)$ and the norm by $|\cdot|$.

For the theory of maximal operators and associated evolution equations we refer the reader to the monograph of Brézis [3]. Recall the important fact that a (nonlinear) version of the Hille–Yosida–Phillips theorem holds for such operators. Recall also that for any compact interval $J = [t_0, t_1]$, any $f \in L^1(J, H)$, and any $u_0 \in \overline{D(A)}$ there exists a unique weak solution on J of

$$(du/dt) + Au \ni f \tag{1}$$

satisfying $u(t_0) = u_0$. A significant class of ordinary and partial differential equations is of form (1).

We shall be concerned here with solutions on the whole of $\mathbb{R}$. So let J be a noncompact interval and let $f \in L^1_{\text{loc}}(J, H)$. The function $u \in C(J, H)$ is said to be a strong (respectively weak) solution of (1) on J if its restriction to any compact interval $J' \subset J$ is a strong (respectively weak) solution of (1) on J'. Clearly, for any interval of the form $J = [t_0, +\infty]$, any $f \in L^1_{\text{loc}}(J, H)$ and any $u_0 \in \overline{D(A)}$ there exists a unique weak solution of (1) on J satisfying $u(t_0) = u_0$.

For any $t \in \mathbb{R}$ denote by κ_t the characteristic function of $]-\infty, t]$. The causal extension $L_e{}^p(\mathbb{R}, H)$, $p \in [1, +\infty]$, is the vector space of (classes of) functions f in $L^1_{\text{loc}}(\mathbb{R}, H)$ such that the causal truncation $\kappa_t f$ of f belongs to $L^p(\mathbb{R}, H)$ for all $t \in \mathbb{R}$.

The main purpose of this paper is to discuss for Eq. (1) a passivity version as defined in systems theory; see, for instance, Dolph [9] and Zemanian [11] for the linear case and Desoer and Vidyasagar [8] for the nonlinear one. Assume $0 \in A0$; Eq. (1) is said to be passive if to any $f \in L_e{}^2(\mathbb{R}, H)$ one may assign a weak solution u of (1) on $\mathbb{R}$ such that $u \in L_e{}^2(\mathbb{R}, H)$ and such that

$$\int_{-\infty}^{t} (f - \hat{f}, u - \hat{u})\, d\sigma \geq 0, \qquad \forall t \in \mathbb{R}, \tag{2}$$

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for any f and $\hat{f}$ in $L_e^2(\mathbb{R}, H)$ and any assigned solutions u, respectively $\hat{u}$. It will be shown that when A is maximal monotone, condition (2) is in fact a consequence of the requirement that to any $f \in L_e^2(\mathbb{R}, H)$ one may assign a weak solution $u \in L_e^2(\mathbb{R}, H)$; moreover, the solution u assigned to f is unique.

It seems of interest in systems theory to discuss the properties implied by passivity. In Section 2 we establish that if Eq. (1) is passive, the operator $f \mapsto u$ commutes with any translation operator and possesses certain continuity properties. Moreover, this operator is causal, which means if u and $\hat{u}$ are assigned to f and $\hat{f}$, respectively, one has $u = \hat{u}$ on $]-\infty, T]$ whenever $f = \hat{f}$ a.e. in $]-\infty, T]$. We also establish a simple passivity criterion. In the above problems an essential point is the maximal monotony of the L^2-operator associated with Eq. (1), which is the reciprocal of $f \mapsto u$.

Note that the more general problem of when $f \in L_e^{p_1}(\mathbb{R}, H)$ implies (1) has a solution $u \in L_e^{p_2}(\mathbb{R}, H)$ also seems to be natural for some choices of p_1 and p_2 other than $p_1 = p_2 = 2$. However, except for the very easy case $p_1 = 1, p_2 = \infty$ (see the end of Section 2), some difficulties arise here. For instance, in the case $p_1 = p_2 \neq 2$, when discussing continuity properties of the operator $f \mapsto u$ we have to consider operators whose reciprocals are m-accretive, and very little seems to be known about such operators.

We say that Eq. (1) is α-strongly passive ($\alpha > 0$) if it is passive with (2) replaced by

$$\int_{-\infty}^{t} (f - \hat{f}, u - \hat{u})\, d\sigma \geqslant \alpha \int_{-\infty}^{t} | u - \hat{u} |^2\, d\sigma, \qquad \forall t \in \mathbb{R}. \tag{3}$$

By using the Cauchy–Schwarz inequality we see that in this case the operator $f \mapsto u$ possesses the Lipschitz property

$$\| \kappa_t(u - \hat{u})\|_{L^2(\mathbb{R},H)} \leqslant \alpha^{-1} \| \kappa_t(f - \hat{f})\|_{L^2(\mathbb{R},H)}, \qquad \forall t \in \mathbb{R}.$$

The maximal monotone operator A is said to be α-strongly monotone if $(y_1 - y_2, x_1 - x_2) \geqslant \alpha \,| x_1 - x_2 |^2$, for all $[x_1, y_1] \in A$ and all $[x_2, y_2] \in A$ ($[x, y] \in A$ stays for $y \in Ax$). In Section 3 we show that Eq. (1) is α-strongly passive if and only if A is α-strongly monotone. This extends to our framework a result established previously for linear evolution equations by Beltrami and Buianouckas [1], by arguments which use the linecrity in an essential way. The proof we give is based on certain results in the theory of nonlinear semigroups of contractions.

In Section 4 we discuss L^p-stability of Eq. (1) (i.e., the problem of when $f \in L^p(\mathbb{R}^+, H)$ implies that all the solutions of (1) on $\mathbb{R}^+$ belong to the same class). For $p \in [1, +\infty[$ we sketch some results which may be obtained by using the ideas we applied to discuss passivity; for $p = \infty$ our main result states that when A is a subdifferential, a strong version of L^∞-stability is equivalent to the coercivity of A.

2. Passivity

We shall make use of the result (see [3]),

Lemma 1. *Assume that A is α-strongly monotone. Let J be an interval of $\mathbb{R}$ and $f, \hat{f}$ be in $L^1_{loc}(J, H)$. If u and $\hat{u}$ are weak solutions on J of*

$$(du/dt) + Au \ni f, \qquad \textit{respectively} \qquad (d\hat{u}/dt) + A\hat{u} \ni \hat{f},$$

we have for any s and t in J, $s \leqslant t$,

$$|u(t) - \hat{u}(t)| \leqslant e^{-\alpha(t-s)} |u(s) - \hat{u}(s)| + \int_s^t e^{-\alpha(t-\sigma)} |f(\sigma) - \hat{f}(\sigma)| \, d\sigma. \tag{4}$$

Denote by $S(\mathbb{R}, H)$ the Stepanov space (the space of (classes of) functions f in $L^1_{loc}(\mathbb{R}, H)$ such that

$$\sup_{t \in \mathbb{R}} \int_t^{t+1} |f(\sigma)| \, d\sigma < \infty,$$

with the above supremum as $\|f\|_{S(\mathbb{R},H)}$). The causal extension of $S(\mathbb{R}, H)$ is denoted by $S_e(\mathbb{R}, H)$. Clearly $L^p(\mathbb{R}, H) \subset S(\mathbb{R}, H)$ and $L_e^p(\mathbb{R}, H) \subset S_e(\mathbb{R}, H)$ for all $p \in [1, +\infty]$.

Lemma 2. *Assume that A is α-strongly monotone and $0 \in A0$. Then*

(i) *for any f in $S_e(\mathbb{R}, H)$ there exists a unique u in $S_e(\mathbb{R}, H)$ such that u is a weak solution of* (1) *on $\mathbb{R}$; this solution belongs to $L_e^\infty(\mathbb{R}, H)$; moreover, if u and $\hat{u}$ are the solutions assigned to f and $\hat{f}$, respectively, we have for all t in $\mathbb{R}$*

$$\begin{aligned} |u(t) - \hat{u}(t)| &\leqslant \int_{-\infty}^t e^{-\alpha(t-\sigma)} |f(\sigma) - \hat{f}(\sigma)| \, d\sigma \\ &\leqslant (1 - e^{-\alpha})^{-1} \|\kappa_t(f - \hat{f})\|_{S(\mathbb{R},H)} \, ; \end{aligned} \tag{5}$$

(ii) *if $f \in L_e^p(\mathbb{R}, H)$ for a $p \in [1, +\infty]$, the corresponding solution u belongs to $L_e^r(\mathbb{R}, H)$ for all $r \in [p, +\infty]$ and, if $p < \infty$, $u(t) \to 0$ as $t \to -\infty$; moreover, when p, r, and q satisfy*

$$p \in [1, +\infty], \qquad r \in [p, +\infty] \qquad \textit{and} \qquad r^{-1} = p^{-1} + q^{-1} - 1,$$

we have for any functions f and $\hat{f}$ in $L_e^p(\mathbb{R}, H)$ and any assigned solutions u and $\hat{u}$, respectively,

$$\|\kappa_t(u - \hat{u})\|_{L^r(\mathbb{R},H)} \leqslant (q\alpha)^{-1/q} \|\kappa_t(f - \hat{f})\|_{L^p(\mathbb{R},H)}, \qquad \forall t \in \mathbb{R},$$

if q is finite and

$$\| \kappa_t(u - \hat{u})\|_{L^\infty(\mathbb{R},H)} \leqslant \| \kappa_t(f - \hat{f})\|_{L^1(\mathbb{R},H)}, \qquad \forall t \in \mathbb{R},$$

if $q = +\infty$ (which is equivalent to $p = 1$, $r = +\infty$).

Proof. Let (s_n) be a decreasing sequence of real numbers, $s_n \to -\infty$, and let u_n be the weak solution on $[s_n, +\infty[$ of

$$(du_n/dt) + Au_n \ni f, \qquad u_n(s_n) = 0.$$

Since the constant function $w = 0$ on $\mathbb{R}$ is a solution of

$$(dw/dt) + Aw \ni 0, \tag{6}$$

we have, by Lemma 1,

$$| u_n(t)| \leqslant \int_{s_n}^t e^{-\alpha(t-\sigma)} | f(\sigma)| \, d\sigma \leqslant \sum_{j=0}^{\infty} \int_{t-(j+1)}^{t-j} e^{-\alpha(t-\sigma)} | f(\sigma)| \, d\sigma,$$

so that

$$| u_n(t)| \leqslant (1 - e^{-\alpha})^{-1} \| \kappa_t f \|_{S(\mathbb{R},H)}, \qquad \text{for all } t \in [s_n, +\infty[\text{ and all } n. \tag{7}$$

Now let $J = [a, b]$ be an arbitrary compact interval. If $s_n \leqslant s_m \leqslant a$, Lemma 1 and (7) yield for all $t \in J$

$$| u_n(t) - u_m(t)| \leqslant e^{-\alpha(t-s_m)} | u_n(s_m)| \leqslant (1 - e^{-\alpha})^{-1} e^{-\alpha(a-s_m)} \| \kappa_a f \|_{S(\mathbb{R},H)},$$

and thus (u_n) is a Cauchy sequence in $C(J, H)$. Since u_n are weak solutions of (1) on J, the limit u of (u_n) possesses the same property. Clearly u may be extended to a weak solution of (1) on the whole of $\mathbb{R}$ and, by (7) $u \in L_e^\infty(\mathbb{R}, H)$.

To prove uniqueness assume that $u \in S_e(\mathbb{R}, H)$ and $v \in S_e(\mathbb{R}, H)$ are two solutions assigned to f. By Lemma 1,

$$| u(t) - v(t)| \leqslant e^{-\alpha(t-\sigma)} | u(\sigma) - v(\sigma)|, \qquad \forall \sigma, t \in \mathbb{R}, \qquad \sigma \leqslant t.$$

An integration with respect to σ on $[s - 1, s]$, $s \leqslant t$, yields

$$| u(t) - v(t)| \leqslant e^{-\alpha(t-s)} \| \kappa_t(u - v)\|_{S(\mathbb{R},H)}, \qquad \forall s, t \in \mathbb{R}, \qquad s \leqslant t,$$

and so $u = v$ follows by letting $s \to -\infty$.

To verify (5), it suffices to note that since u and $\hat{u}$ belong to $L_e^\infty(\mathbb{R}, H)$, (4) yields as $s \to -\infty$

$$\begin{aligned} | u(t) - \hat{u}(t)| &\leqslant \int_{-\infty}^t e^{-\alpha(t-\sigma)} | f(\sigma) - \hat{f}(\sigma)| \, d\sigma \\ &= \sum_{j=0}^{\infty} \int_{t-(j+1)}^{t-j} e^{-\alpha(t-\sigma)} | f(\sigma) - \hat{f}(\sigma)| \, d\sigma. \end{aligned}$$

This completes the proof of the assertions in (i).

We now prove the assertions in (ii). By applying (5) to u and $w = 0$, we deduce

$$| u(t)| \leqslant \int_{-\infty}^{t} e^{-\alpha(t-\sigma)} \,| f(\sigma)| \, d\sigma, \qquad \forall t \in R.$$

The function

$$t \mapsto F(t) = \int_{-\infty}^{t} e^{-\alpha(t-\sigma)} \,| f(\sigma)| \, d\sigma$$

is the convolution of $|f|$ with the function G defined by $G(t) = 0$, if $t < 0$, $G(t) = e^{-\alpha t}$, if $t \geqslant 0$. Since $G \in L^s(\mathbb{R})$ for all $s \in [1, +\infty]$ and since $|f| \in L_e{}^p(\mathbb{R})$, the function F belongs to $L_e{}^r(\mathbb{R})$ (see [10, VI.11.7]) and so $u \in L_e{}^r(\mathbb{R}, H)$ for all $r \in [p, +\infty]$. The other claims in (ii) are now straightforward.

Remark 1. If f is Stepanov almost periodic, the corresponding solution u is Bohr almost periodic (for almost periodic functions we refer the reader to [5]). This follows easily by (5), since for any $\tau \in \mathbb{R}$, $t \mapsto u(t + \tau)$ is the solution assigned to the function $t \mapsto f(t + \tau)$. The existence and uniqueness of bounded solutions which are Bohr almost periodic when f is Stepanov almost periodic was essentially established in a previous paper by Biroli [2], using arguments that are similar enough to ours.

We now discuss passivity. Assume $0 \in A0$. For any $T \in R$ define the (multivalued) operator $\mathscr{B}_T$ of $L^2(-\infty, T, H)$ by $f \in \mathscr{B}_T u$ if u is a weak solution of (1) on $[-\infty, T]$.

THEOREM 1. *For any $T \in \mathbb{R}$ the operator $\mathscr{B}_T$ is maximal monotone and $\mathscr{B}_T^{-1}$ is single valued on its domain $D(\mathscr{B}_T^{-1})$. Moreover, for any $f \in D(\mathscr{B}_T^{-1})$, $\mathscr{B}_T^{-1}f(t) \to 0$ as $t \to -\infty$ and $\mathscr{B}_T^{-1}f \in L^r(-\infty, T, H)$ for all $r \in [2, +\infty]$. If there exists $T_0 \in \mathbb{R}$ such that $\mathscr{B}_{T_0}$ is surjective, then for all $T \in \mathbb{R}$, $\mathscr{B}_T$ is surjective, $\mathscr{B}_T^{-1}$ is demicontinuous, and the operator $f \mapsto \mathscr{B}_T^{-1}f$ from $L^2(-\infty, T, H)$ to $L^r(-\infty, T, H)$ is locally Holder of order $2^{-1} - r^{-1}$ for any $r \in]2, +\infty]$.*

Proof. Assume $u \in \mathscr{B}_T^{-1}f$, i.e., u is a weak solution of (1) on $]-\infty, T]$. Extend f and u to $\tilde{f}$ and $\tilde{u}$, respectively, by $\tilde{f}(t) = 0$ if $t > T$ and $\tilde{u}$ is on $[T, +\infty[$, the weak solution of

$$(d\tilde{u}/dt) + A\tilde{u} \ni \tilde{f}, \qquad \tilde{u}(T) = u(T).$$

Clearly $\tilde{f}$ and $\tilde{u}$ belong to $L_e{}^2(\mathbb{R}, H)$ and $\tilde{u}$ is a weak solution of

$$(d\tilde{u}/dt) + A\tilde{u} \ni \tilde{f}$$

on the whole of $\mathbb{R}$. Then $\tilde{u}$ is a weak solution on $\mathbb{R}$ of

$$(d\tilde{u}/dt) + (A + I)\, \tilde{u} \ni \tilde{f} + \tilde{u},$$

where I is the identity on H. Since $A + I$ is strongly monotone, it follows, by

Lemma 2, that $\tilde{u}(t) \to 0$ as $t \to -\infty$ and $\tilde{u} \in L_e^r(\mathbb{R}, H)$ for all $r \in]2, +\infty]$. Hence $u(t) \to 0$ as $t \to -\infty$ and $u \in L^r(-\infty, T, H)$ for all $r \in]2, +\infty]$.

Assume now that u and $\hat{u}$ belong to $\mathscr{B}_T^{-1}f$, i.e., u and $\hat{u}$ are weak solutions of (1) on $]-\infty, T]$. Then according to [3, Lemma 3.1] we have

$$|u(t) - \hat{u}(t)| \leqslant |u(s) - \hat{u}(s)| \qquad \text{for all } t \text{ and } s \text{ in }]-\infty, T],\ s \leqslant t.$$

Since $u(s)$ and $\hat{u}(s)$ tend to 0 as $s \to -\infty$, we deduce $u = \hat{u}$ and so $\mathscr{B}_T^{-1}$ is single valued on its domain.

Assume $u \in \mathscr{B}_T f$ and $\hat{u} \in \mathscr{B}_T \hat{f}$. According to [3, Chap. III, Sect. 2] the monotony of A implies, for all $s \in]-\infty, T]$

$$\int_s^T (f - \hat{f}, u - \hat{u})\, d\sigma \geqslant \tfrac{1}{2} |u(T) - \hat{u}(T)|^2 - \tfrac{1}{2} |u(s) - \hat{u}(s)|^2.$$

By letting $s \to -\infty$ we see that $\mathscr{B}_T$ is monotone. Then, according to [3, Proposition 2.2], to prove maximality it suffices to verify that the range of $\mathscr{I} + \mathscr{B}_T$ is $L^2(-\infty, T, H)$ ($\mathscr{I}$ is the identity on $L^2(-\infty, T, H)$). But $f \in (\mathscr{I} + \mathscr{B}_T)\, u$ if and only if the following three conditions are satisfied: f and u belong to $L^2(-\infty, T, H)$ and u is a weak solution on $]-\infty, T]$ of

$$(du/dt) + (A + I)\, u \ni f.$$

By considering the extension $\tilde{f}$ of f and applying Lemma 2, we see that the range of $\mathscr{I} + \mathscr{B}_T$ is $L^2(-\infty, T, H)$.

In order to prove the last assertions of Theorem 1 we establish that

$$\| \mathscr{B}_T^{-1}f - \mathscr{B}_T^{-1}g \|_{L^r(-\infty,T,H)} \leqslant (\tfrac{1}{2} - (1/r))^{(1/r)-(1/2)} \| \mathscr{B}_T^{-1}f - \mathscr{B}_T^{-1}g \|_{L^2(-\infty,T,H)}^{(1/2)+(1/r)} \| f - g \|_{L^2(-\infty,T,H)}^{(1/2)-(1/r)} \tag{8}$$

for all $T \in \mathbb{R}$, all $r \in]2, +\infty]$, and all f, g in $D(\mathscr{B}_T^{-1})$. To prove (8), put $u = \mathscr{B}_T^{-1}f$ and $v = \mathscr{B}_T^{-1}g$. Clearly (8) holds if $f = g$ or $u = v$. So, assume $f \neq g$ and $u \neq v$. Extend f and g to $\tilde{f}$ and $\tilde{g}$, respectively, by $\tilde{f}(t) = \tilde{g}(t) = 0$ if $t > T$. Extend u and v to $\tilde{u}$ and $\tilde{v}$, respectively, which are defined on $[T, +\infty[$ as the weak solutions of

$$(d\tilde{u}/dt) + A\tilde{u} \ni \tilde{f},\ \ \tilde{u}(T) = u(T), \quad \text{respectively,} \quad (d\tilde{v}/dt) + A\tilde{v} \ni \tilde{g},\ \ \tilde{v}(T) = v(T).$$

Then $\tilde{u}, \tilde{v}, \tilde{f}$, and $\tilde{g}$ belong to $L_e^2(\mathbb{R}, H)$ and, for any $\alpha > 0$, $\tilde{u}$ and $\tilde{v}$ are weak solutions on $\mathbb{R}$ of

$$(d\tilde{u}/dt) + (A + \alpha I)\, \tilde{u} \ni \tilde{f} + \alpha\tilde{u}, \quad \text{respectively,} \quad (d\tilde{v}/dt) + (A + \alpha I)\, \tilde{v} \ni \tilde{g} + \alpha\tilde{v}.$$

By applying Lemma 2,

$$\| u - v \|_{L^r(-\infty,T,H)} = \| \kappa_T(\tilde{u} - \tilde{v})\|_{L^r(\mathbb{R},H)}$$
$$\leqslant q^{-1/q}\{\alpha^{-(1/q)} \| f - g \|_{L^2(-\infty,T,H)} + \alpha^{1-(1/q)} \| u - v \|_{L^2(-\infty,T,H)}\},$$

where $q = 2 - 4(r + 2)^{-1}$. The best estimation is obtained for

$$\alpha = (q - 1)^{-1} \| f - g \|_{L^2(-\infty,T,H)} \| u - v \|^{-1}_{L^2(-\infty,T,H)}$$

and (8) follows for this value of α.

Finally, assume that $\mathscr{B}_{T_0}$ is surjective. Then, it is easy to see by extension and restriction that $\mathscr{B}_T$ is surjective for all $T \in \mathbb{R}$. Hence $\mathscr{B}_T^{-1}$ is demicontinuous and locally bounded (see [3, Corollary 2.5 and Theorem 2.3]). It follows then by (8) that the operator $f \mapsto \mathscr{B}_T^{-1} f$ from $L^2(-\infty, T, H)$ to $L^r(-\infty, T, H)$ is locally Holder of order $2^{-1} - r^{-1}$.

The following corollaries of Theorem 1 are straightforward by extension and restriction.

Corollary 1. *Equation* (1) *is passive if and only if there exists* $T_0 \in \mathbb{R}$ *such that* $\mathscr{B}_{T_0}$ *is surjective.*

Corollary 2. *Assume Eq.* (1) *is passive. Then to any* f *in* $L_e^2(\mathbb{R}, H)$ *one can assign a unique weak solution* u *in* $L_e^2(\mathbb{R}, H)$ *of* (1) *on* $\mathbb{R}$; $u(t) \to 0$ *as* $t \to -\infty$ *and, for all* $T \in \mathbb{R}$, *the restriction of* u *to* $]-\infty, T]$ *is the image by* $\mathscr{B}_T^{-1}$ *of the restriction of* f *to* $]-\infty, T]$. *Moreover, if* u_n *and* u *are the solutions assigned to* f_n *and* f, *respectively, and if the sequence* $\kappa_T f_n \to \kappa_T f$ *in* $L^2(\mathbb{R}, H)$, *then* $\kappa_T u_n \to \kappa_T u$ *weakly in* $L^2(\mathbb{R}, H)$ *and strongly in* $L^r(\mathbb{R}, H)$, *for all* $r \in]2, +\infty]$.

From Corollary 2 we deduce

Corollary 3. *Assume Eq.* (1) *is passive. Then the operator* $f \mapsto u$ *that assigns to* $f \in L_e^2(\mathbb{R}, H)$ *the weak solution* $u \in L_e^2(\mathbb{R}, H)$ *of* (1) *on* $\mathbb{R}$ *is causal and commutes with any translation operator.*

Under stronger assumptions on A or f the solution u assigned to f is more regular:

Proposition 1. *Assume that* A *is the subdifferential of a function* φ: $H \to]-\infty, +\infty]$, *which is proper, convex, lower semicontinuous, and satisfies*

$$\inf\{\varphi(x) : x \in H\} = \varphi(0) = 0. \tag{9}$$

In addition to this assume the Eq. (1) *passive. Then, for any* $f \in L_e^2(\mathbb{R}, H)$ *the solution* $u \in L_e^2(\mathbb{R}, H)$ *of* (1) *on* $\mathbb{R}$ *is a strong one,*

$$(du/dt) \in L_e^2(\mathbb{R}, H) \quad \textit{and} \quad \| \kappa_t(du/dt)\|_{L^2(\mathbb{R},H)} \leqslant \| \kappa_t f \|_{L^2(\mathbb{R},H)}, \qquad \forall t \in \mathbb{R}; \tag{10}$$

moreover, when $f \in W_e^{1,2}(\mathbb{R}, H)$ *we have for all* $r \in]2, +\infty]$, $u \in W_e^{1,r}(\mathbb{R}, H)$ *and*

$$\| \kappa_t(du/dt)\|_{L^r(\mathbb{R},H)} \leqslant (\tfrac{1}{2} - (1/r))^{(1/r)-(1/2)} \| \kappa_t f \|_{L^2(\mathbb{R},H)}^{(1/2)+(1/r)} \| \kappa_t(df/dt)\|_{L^2(\mathbb{R},H)}^{(1/2)-(1/r)} \quad \forall t \in \mathbb{R}. \tag{11}$$

Proof. According to [3, Theorem 3.6], u is a strong solution of (1) on $\mathbb{R}$; moreover, for any $s, t \in \mathbb{R}$, $s < t$ and any $\delta \in]0, t - s]$ the restriction of du/dt to $[s + \delta, t]$ belongs to $L^2(s + \delta, t, H)$ and satisfies

$$\left(\int_{s+\delta}^t | du/dt |^2 \, d\sigma\right)^{1/2} \leqslant \left(\int_s^t | f |^2 \, d\sigma\right)^{1/2} + (2\delta)^{-(1/2)} \int_s^{s+\delta} | f | \, d\sigma + (2\delta)^{-(1/2)} | u(s)|.$$

Then (10) follows by letting $s \to -\infty$.

Assume $f \in W_e^{1,2}(\mathbb{R}, H)$, $r \in]2, +\infty]$ and let $t \in \mathbb{R}$, $h > 0$ be arbitrary. Denote by τ_{-h} the translation by h. By using Corollaries 2 and 3 combined with (8), we see that

$$\begin{aligned} &\| \tau_{-h} u - u \|_{L^r(-\infty, t-h, H)} \\ &\quad \leqslant (\tfrac{1}{2} - (1/r))^{(1/r)-(1/2)} \left(\int_{-\infty}^{t-h} | u(\sigma + h) - u(\sigma)|^2 \, d\sigma\right)^{(1/4)+(1/2r)} \\ &\quad \times \left(\int_{-\infty}^{t-h} | f(\sigma + h) - f(\sigma)|^2 \, d\sigma\right)^{(1/4)-(1/2r)}. \end{aligned}$$

Then (11) follows easily by [3, Appendix], Theorem 1, and (10), since u and f belong to $W_e^{1,2}(\mathbb{R}, H)$.

PROPOSITION 2. *Assume Eq.* (1) *is passive and let* f *belong to* $L_e^2(\mathbb{R}, H)$. *If, in addition to this,* $f \in B\, V_{\text{loc}}(\mathbb{R}, H)$ *and if the variation* $V(s, t)$ *of* f *on* $[s, t]$ *satisfies*

$$V_t = \sup\{V(s, t) : s \leqslant t\} < \infty, \qquad \forall t \in \mathbb{R}, \tag{12}$$

then the solution $u \in L_e^2(\mathbb{R}, H)$ *of* (1) *on* $\mathbb{R}$ *is a strong one,* $u(t) \in D(A)$ *for all* $t \in \mathbb{R}$, u *is differentiable on the right on* $\mathbb{R}$, $(du/dt) \in L_e^\infty(\mathbb{R}, H)$ *and*

$$|(d^+u/dt)(t)| \leqslant V(t, t + 0) + V_t, \qquad \forall t \in \mathbb{R}. \tag{13}$$

Proof. Let t_0 and h be arbitrary in $\mathbb{R}$, $h > 0$. By Corollary 3, the function $t \mapsto \hat{u}(t) = u(t + h)$ is the solution of

$$(d\hat{u}/dt) + A\hat{u} \ni \hat{f},$$

where $\hat{f}(t) = f(t + h)$. Then by [3, Lemma 3.1], we have

$$| u(t_0 + h) - u(t_0)| \leqslant | u(s + h) - u(s)| + \int_s^{t_0} | f(\sigma + h) - f(\sigma)| \, d\sigma, \quad \forall s \leqslant t_0.$$

Since $u(t) \to 0$ as $t \to -\infty$ and since, by applying [3, Proposition A.5]

$$\int_s^{t_0} |f(\sigma + h) - f(\sigma)| \, d\sigma \leqslant hV(s, t_0 + h),$$

we deduce

$$| u(t_0 + h) - u(t_0)| \leqslant h \sup\{V(s, t_0 + h) : s \leqslant t_0\}, \qquad \text{for all } t_0 \text{ and } h \text{ in } \mathbb{R},\ h > 0. \tag{14}$$

Hence u is Lipschitz on any interval of the form $]-\infty, T]$ and the conclusions of Proposition 2 follow by applying [3, Proposition 3.3]; the estimation (13) follows by (14).

Remark 2. One can show that when A is α-strongly monotone with $0 \in A0$ (so that by Lemma 2 and Corollary 1 Eq. (1) is passive), condition (12) on f in Proposition 2 may be weakened as follows: if (12) is replaced by

$$V_t' = \sup\{V(\sigma - 1, \sigma) : \sigma \leqslant t\} < \infty, \qquad \forall t \in \mathbb{R},$$

then all of the conclusions of Proposition 2 hold with (13) replaced by

$$\begin{aligned} |(d^+u/dt)(t)| &\leqslant V(t, t + 0) + \lim_{s \to -\infty} \int_s^t e^{-\alpha(t-\sigma)} d_\sigma V(s, \sigma) \\ &\leqslant V(t, t + 0) + V_t'(1 - e^{-\alpha})^{-1}, \qquad \forall t \in \mathbb{R}. \end{aligned}$$

We now discuss sufficient passivity conditions. As noted above, if A is strongly monotone with $0 \in A0$, Eq. (1) is passive. We show in the theorem below that passivity is still ensured when the strong monotony condition is weakened as follows. There exist $\alpha > 0$ and $\rho > 0$ such that

$$(x, y) \geqslant \alpha \, | \, x \, |^2, \qquad \text{for all } [x, y] \in A \text{ with } | \, x \, | < \rho. \tag{15}$$

This suggests that the passivity property depends mainly on the behavior of A in a neighborhood of 0.

THEOREM 2. *If A satisfies* (15) *and* $0 \in A0$, *Eq.* (1) *is passive.*

Proof. By Corollary 1, it suffices to prove that $\mathscr{B}_0$ is surjective. So let f be arbitrary in $L^2(\mathbb{R}^-, H)$. For any $s < 0$, denote by u_s the function defined as follows: on $]-\infty, s]$, $u_s = 0$ and on $[s, 0]$ u_s is the weak solution of

$$(du/dt) + Au \ni f, \qquad u_s(s) = 0.$$

Clearly $u_s = \mathscr{B}_0^{-1}(1 - \kappa_s)f$. It suffices to establish the existence of a $\tau < 0$, such that the set $\{u_s : s < \tau\}$ is bounded in $L^2(\mathbb{R}^-, H)$; for if this set is bounded, we may apply [7, Lemma 2.3] to the maximal monotone operator $\mathscr{B}_0^{-1}$ and to

$((1-\kappa_s)f)_{s<\tau}$, which tends to f in $L^2(\mathbb{R}^-, H)$ as $s \to -\infty$, to see that $f \in D(\mathscr{B}_0^{-1})$.

Choose $\tau < 0$ such that

$$\left\{\int_{-\infty}^{\tau} |f|^2 \, d\sigma\right\}^{1/2} \leqslant 2^{-1/2}\alpha^{1/2}\rho.$$

We claim that

$$|u_s(t)| < \rho, \qquad \text{for all } s < \tau \text{ and all } t \leqslant \tau. \tag{16}$$

Indeed, assume $s < \tau$ and put

$$\theta = \sup\{\sigma \in [s, \tau] : |u_s(t)| < \rho, \text{ for all } t \in [s, \sigma]\}.$$

Since u_s is continuous and $u_s(s) = 0$, we have $\theta > s$. As in the proof of [3, Lemma 3.1], the conditions $0 \in A0$, (15) and $|u_s| < \rho$ on $[s, \theta[$ imply

$$|u_s(t)| \leqslant \int_s^t e^{-\alpha(t-\sigma)} |f(\sigma)| \, d\sigma, \qquad \text{for all } t \in [s, \theta[,$$

hence we have for all $t \in [s, \theta]$

$$|u_s(t)| \leqslant \int_{-\infty}^t e^{-\alpha(t-\sigma)} |f(\sigma)| \, d\sigma \leqslant (2\alpha)^{-1/2} \left\{\int_{-\infty}^t |f|^2 \, d\sigma\right\}^{1/2} \leqslant 2^{-1}\rho. \tag{17}$$

Since u_s is continuous at θ, we deduce $\theta = \tau$, and so (16) follows by (17).

Since (17) holds for all $s < \tau$ and all $t \leqslant \tau$, we have

$$\left\{\int_{-\infty}^{\tau} |u_s|^2 \, d\sigma\right\}^{1/2} \leqslant \alpha^{-1} \left\{\int_{-\infty}^{\tau} |f|^2 \, d\sigma\right\}^{1/2} \qquad \text{for all } s < \tau. \tag{18}$$

By applying [3, Lemma 3.1] combined with (16), we have for all $s < \tau$ and all $t \in [\tau, 0]$

$$|u_s(t)| \leqslant |u_s(\tau)| + \int_\tau^t |f| \, d\sigma \leqslant \rho + |\tau|^{1/2} \left\{\int_\tau^0 |f|^2 \, d\sigma\right\}^{1/2}. \tag{19}$$

Clearly (18) and (19) imply that the set $\{u_s\colon s < \tau\}$ is bounded in $L^2(\mathbb{R}^-, H)$.

Remark 3. Assume that A is the subdifferential of a function φ: $H \to]-\infty, +\infty]$, which is proper, convex, lower, semicontinuous, and satisfies (9). If there exist $\alpha > 0$ and $\rho > 0$ such that

$$\varphi(x) \geqslant \alpha |x|^2, \qquad \text{for all } x \in H \text{ with } |x| < \rho, \tag{20}$$

condition (15) holds. As a matter of fact, $[x, y] \in A$, $|x| < \rho$, implies

$$(x, y) = \varphi(x) + \psi(y) \geqslant \alpha |x|^2,$$

since by (9) the conjugate to φ function ψ is positive on H. The above conditions on φ are satisfied if, for instance, φ is the norm of H (so that $Ax = |x|^{-1}x$, if $x \neq 0$ and $A0$ is the closed unit ball of H with center 0).

Note that Theorem 2 furnishes some perturbation results. For instance, assume that A satisfies conditions in Theorem 2 and let B be a maximal monotone operator such that $0 \in B0$ and such that either $D(A) \cap (\operatorname{Int} D(B))$ or $(\operatorname{Int} D(A)) \cap D(B)$ is nonvoid. Then, according to 3, Corollary 2.7] $A + B$ is maximal monotone and since it satisfies (15) and $0 \in (A + B)\,0$, the equation

$$(du/dt) + (A + B)\,u \ni f$$

is also passive.

Remark 4. Under the assumptions of Theorem 2, the operator $f \mapsto u$ possesses the following additional continuity property: if $u_n \in L_e^2(\mathbb{R}, H)$ and $u \in L_e^2(\mathbb{R}, H)$ are the solutions of (1) on $\mathbb{R}$ assigned to $f_n \in L_e^2(\mathbb{R}, H)$ and $f \in L_e^2(\mathbb{R}, H)$, respectively, and if the sequence $\kappa_T f_n \to \kappa_T f$ in $L^2(\mathbb{R}, H)$, then $\kappa_T u_n \to \kappa_T u$ (strongly) in $L^2(\mathbb{R}, H)$. This follows easily by (15) combined with the fact that, according to Corollary 2, $\kappa_T u_n \to \kappa_T u$ in $L^\infty(\mathbb{R}, H)$ and $u(t) \to 0$ as $t \to -\infty$.

We conclude this section by noting that under only the assumptions that A is maximal monotone and $0 \in A$, the following L^1-passivity property holds: for any $f \in L_e^1(\mathbb{R}, H)$ there exists a unique u such that u is a weak solution on $\mathbb{R}$ of

$$(du/dt) + Au \ni f, \qquad \lim_{t \to -\infty} u(t) = 0;$$

moreover, the operator $f \mapsto u$ is causal and, for any solutions u and $\hat{u}$ assigned to f and $\hat{f}$, respectively, (2) holds and

$$|u(t) - \hat{u}(t)| \leqslant \int_{-\infty}^{t} |f - \hat{f}|\, d\sigma, \qquad \text{for all } t \in \mathbb{R}.$$

The existence may be proved by applying [3, Lemma 3.1] to (u_n) defined in the proof of Lemma 2. Then the other assertions are straightforward.

3. Strong Passivity

For any $\lambda > 0$ denote by J_λ the (nonlinear) resolvent $(I + \lambda A)^{-1}$ of A.

THEOREM 3. *Assume $0 \in A0$ and let α be a strictly positive number. Then the following three assertions are equivalent:*

(i) *Eq.* (1) *is α-strongly passive;*

(ii) *A is α-strongly monotone;*

(iii) *for all* $\lambda > 0$, *all* $y_1 \in H$ *and all* $y_2 \in H$,

$$(J_\lambda y_1 - J_\lambda y_2 , y_1 - y_2) \geqslant \lambda\alpha \mid J_\lambda y_1 - J_\lambda y_2 \mid^2.$$

Proof of (ii) $\Rightarrow$ (iii). Let y_i , $i = 1, 2$ be arbitrary in H and put $x_i = J_\lambda y_i$, hence $\lambda^{-1}(y_i - x_i) \in Ax_i$. Then (ii) implies

$$\begin{aligned}(x_1 - x_2 , y_1 - y_2) &= \lambda(x_1 - x_2 , \lambda^{-1}(y_1 - x_1) - \lambda^{-1}(y_2 - x_2)) + \mid x_1 - x_2 \mid^2 \\ &\geqslant \lambda\alpha \mid x_1 - x_2 \mid^2.\end{aligned}$$

Proof of (iii) $\Rightarrow$ (ii). Let $[x_i , y_i]$, $i = 1, 2$ be arbitrary in A and put $v_i = x_i + \lambda y_i$, hence $x_i = J_\lambda v_i$. Then (iii) implies that for all $\lambda > 0$

$$\alpha \mid x_1 - x_2 \mid^2 \leqslant \lambda^{-1}(x_1 - x_2 , v_1 - v_2) = \lambda^{-1} \mid x_1 - x_2 \mid^2 + (x_1 - x_2 , y_1 - y_2),$$

and (ii) follows as $\lambda \to +\infty$.

Proof of (ii) $\Rightarrow$ (i). Since passivity follows by Lemma 2 and Corollary 1, we have only the verify (3).

Assume first that u and $\hat{u}$ are strong solutions on $[s, t]$. Since A is assumed α-strongly monotone we have

$$(f - (du/dt) - \hat{f} + (d\hat{u}/dt), u - \hat{u}) \geqslant \alpha \mid u - \hat{u} \mid^2, \qquad \text{a.e. in } [s, t],$$

and so, by integration,

$$\int_s^t (f - \hat{f}, u - \hat{u})\, d\sigma \geqslant \alpha \int_s^t \mid u - \hat{u} \mid^2 d\sigma + \tfrac{1}{2} \mid u(t) - \hat{u}(t)\mid^2 - \tfrac{1}{2} \mid u(s) - \hat{u}(s)\mid^2. \tag{21}$$

Clearly (21) still holds when u and $\hat{u}$ are weak solutions on $[s, t]$, and so (3) follows by letting $s \to -\infty$, since, according to Corollary 2, $u(s)$ and $\hat{u}(s)$ tend to 0.

In the proof of the last implication of Theorem 3 we make use of

LEMMA 3. *Let T be a strictly positive number*, (f_n) *be a sequence of members of* $L^1(0, T, H)$, (z_{0n}) *a sequence of elements in* $\overline{D(A)}$, and (r_n) *a sequence of strictly positive numbers. Let z_n be the weak solution on* $[0, T]$ *of*

$$(dz_n/dt) + r_n A z_n \ni f_n , \qquad z_n(0) = z_{0n} .$$

Then if $f_n \to f$ *in* $L^1(0, T, H)$, *if* $z_{0n} \to z_0$ *and if* $r_n \to 0$, *the sequence* (z_n) *converges in* $C([0, T], H)$ *to the weak solution on* $[0, T]$ *of*

$$(dz/dt) + \partial\Phi(z) \ni f, \qquad z(0) = z_0 ,$$

where $\partial\Phi$ is the subdifferential of the indicator function Φ of $\overline{D(A)}$.

Proof of Lemma 3. According to [3, Theorem 3.16], it suffices to verify that we have a.e. in $[0, T]$

$$(I + \lambda(r_n A - f(t)))^{-1} \zeta \to (I + \lambda(\partial\Phi - f(t))^{-1} \zeta, \qquad \text{as} \quad n \to \infty, \tag{22}$$

for all $\lambda > 0$ and all $\zeta \in D(A)$.

Since

$$(I + \lambda(r_n A - f(t)))^{-1} \zeta = (I + r_n \lambda A)^{-1} (\zeta + \lambda f(t)),$$

we have by [3, Theorem 2.2]

$$(I + \lambda(r_n A - f(t)))^{-1} \zeta \to \operatorname{Proj}_{\overline{D(A)}}(\zeta + \lambda f(t)), \qquad \text{as} \quad n \to \infty.$$

Since Φ is the indicator function of $\overline{D(A)}$, the resolvent of $\partial\Phi$ is the projection on $\overline{D(A)}$. Hence

$$(I + \lambda(\partial\Phi - f(t)))^{-1} \zeta = (I + \lambda\, \partial\Phi)^{-1}(\zeta + \lambda f(t)) = \operatorname{Proj}_{\overline{D(A)}}(\zeta + \lambda f(t))$$

so that (22) follows.

Proof of the implication (i) $\Rightarrow$ (ii) *of Theorem* 3. Denote by S the semigroup generated by $-A$ on $\overline{D(A)}$. It suffices to prove that

$$| S(t)\, u_0 - S(t)\, \hat{u}_0 | \leqslant e^{-\alpha t} | u_0 - \hat{u}_0 |, \quad \forall u_0 \in D(A), \quad \forall \hat{u}_0 \in D(A), \quad \forall t > 0, \tag{23}$$

for, as noted in [6], (23) implies that $A - \alpha I$ is monotone, hence A is α-strongly monotone.

To prove (23), consider the approximation $(\psi_n)_{n \geqslant 1}$ of the Dirac measure defined by $\psi_n(t) = n$, if $t \in [0, n^{-1}]$, $\psi_n(t) = 0$, if $t \notin [0, n^{-1}]$. Fix an arbitrary $s > 0$ and define f_n and $\hat{f}_n$ on $\mathbb{R}$ by

$$f_n(t) = \psi_n(t)\, u_0 - \psi_n(t - s)\, S(s)\, u_0,$$

respectively

$$\hat{f}_n(t) = \psi_n(t)\, \hat{u}_0 - \psi_n(t - s)\, S(s)\, \hat{u}_0.$$

Denote by u_n and $\hat{u}_n$ the solutions in $L_e^2(\mathbb{R}, H)$ of

$$(du_n/dt) + Au_n \ni f_n\,, \qquad \text{respectively} \qquad (d\hat{u}_n/dt) + A\hat{u}_n \ni \hat{f}_n\,,$$

the existence of which is ensured by (i).

Causality implies

$$u_n(t) = \hat{u}_n(t) = 0 \qquad \text{for all} \quad t \leqslant 0. \tag{24}$$

Then by [3, Proposition 3.3], on $\mathbb{R}^+$ u_n and $\hat{u}_n$ are strong solutions of

$$(du_n/dt) + Au_n \ni f_n, \qquad u_n(0) = 0,$$

respectively

$$(d\hat{u}_n/dt) + A\hat{u}_n \ni \hat{f}_n, \qquad \hat{u}_n(0) = 0.$$

Hence, for sufficiently large n (such that $n^{-1} < s$),

$$\begin{aligned} u_n(t) &= v_n(t), \qquad \text{if} \quad t \in [0, n^{-1}], \\ &= S(t - n^{-1})\, v_n(n^{-1}), \qquad \text{if} \quad t \in]n^{-1}, s], \\ &= w_n(t), \qquad \text{if} \quad t \in]s, s + n^{-1}], \\ &= S(t - s - n^{-1})\, w_n(s + n^{-1}), \qquad \text{if} \quad t > s + n^{-1}, \end{aligned}$$

where v_n is the strong solution on $\mathbb{R}^+$ of

$$(dv_n/dt) + Av_n \ni nu_0, \qquad v_n(0) = 0,$$

and w_n is the strong solution on $[s, +\infty[$ of

$$(dw_n/dt) + Aw_n \ni -n\, S(s)\, u_0, \qquad w_n(s) = S(s - n^{-1})\, v_n(n^{-1}).$$

Let x_n and y_n be the strong solutions on $\mathbb{R}^+$ of

$$(dx_n/dt) + n^{-1} Ax_n \ni u_0, \qquad x_n(0) = 0,$$

respectively

$$(dy_n/dt) + n^{-1} Ay_n \ni -S(s)\, u_0, \qquad y_n(0) = S(s - n^{-1})\, x_n(1).$$

Clearly, for all $t \geqslant 0$, $v_n(n^{-1}t) = x_n(t)$ and $w_n(n^{-1}t + s) = y_n(t)$, hence

$$\begin{aligned} u_n(t) &= x_n(nt), \qquad \text{if} \quad t \in [0, n^{-1}], \\ &= S(t - n^{-1})\, x_n(1), \qquad \text{if} \quad t \in]n^{-1}, s], \\ &= y_n(n(t - s)), \qquad \text{if} \quad t \in]s, s + n^{-1}], \\ &= S(t - s - n^{-1})\, y_n(1), \qquad \text{if} \quad t > s + n^{-1}. \end{aligned}$$

In the same way,

$$\begin{aligned} \hat{u}_n(t) &= \hat{x}_n(nt), \qquad \text{if} \quad t \in [0, n^{-1}], \\ &= S(t - n^{-1})\, \hat{x}_n(1), \qquad \text{if} \quad t \in]n^{-1}, s], \\ &= \hat{y}_n(n(t - s)), \qquad \text{if} \quad t \in]s, s + n^{-1}], \\ &= S(t - s - n^{-1})\, \hat{y}_n(1), \qquad \text{if} \quad t > s + n^{-1}, \end{aligned}$$

where $\hat{x}_n$ and $\hat{y}_n$ are the strong solutions on $\mathbb{R}^+$ of

$$(d\hat{x}_n/dt) + n^{-1}A\hat{x}_n \ni \hat{u}_0\,, \qquad \hat{x}_n(0) = 0,$$

respectively

$$(d\hat{y}_n/dt) + n^{-1}A\hat{y}_n \ni -S(s)\,\hat{u}_0\,, \qquad \hat{y}_n(0) = S(s - n^{-1})\,\hat{x}_n(1).$$

The strong passivity condition (3) and (24) yield

$$\int_0^{s+n^{-1}} (f_n - \hat{f}_n\,, u_n - \hat{u}_n)\, d\sigma \geqslant \alpha \int_0^{s+n^{-1}} |\, u_n - \hat{u}_n\,|^2\, d\sigma, \qquad \text{for all } r. \tag{25}$$

We shall let $n \to \infty$. Note first that

$$\int_0^{s+n^{-1}} (f_n - \hat{f}_n\,, u_n - \hat{u}_n)\, d\sigma$$
$$= \left(u_0 - \hat{u}_0\,, \int_0^1 (x_n - \hat{x}_n)\, d\sigma\right) - \left(S(s)u_0 - S(s)\hat{u}_0\,, \int_0^1 (y_n - \hat{y}_n)\, d\sigma\right).$$

According to Lemma 3, (x_n) converges uniformly on $[0, 1]$ to the solution x of

$$(dx/dt) + \partial\Phi(x) \ni u_0\,, \qquad x(0) = 0.$$

Since $\overline{D(A)}$ is convex, $u_0 \in D(A)$ and $0 \in D(A)$, we have $x(t) = tu_0$ on $[0, 1]$. In the same way, $\hat{x}_n(t) \to t\hat{u}_0$, $y_n(t) \to (1 - t)\,S(s)\,u_0$, and $\hat{y}_n(t) \to (1 - t)\,S(s)\,\hat{u}_0$ uniformly on $[0, 1]$ as $n \to \infty$. Hence

$$\int_0^{s+n^{-1}} (f_n - \hat{f}_n\,, u_n - \hat{u}_n)\, d\sigma \to \tfrac{1}{2}\,|\, u_0 - \hat{u}_0\,|^2 - \tfrac{1}{2}\,|\, S(s)u_0 - S(s)\hat{u}_0\,|^2, \qquad \text{as} \quad n \to \infty.$$

Next, $u_n(t) \to S(t)\,u_0$ on $]0, s]$, $u_n(t) \to 0$ on $]s, +\infty]$, $\hat{u}_n(t) \to S(t)\,\hat{u}_0$ on $]0, s]$, and $\hat{u}_n(t) \to 0$ on $]s, +\infty]$. Since the constant function $w = 0$ is a solution of (6), by applying [3, Lemma 3.1], we see that for all $t \geqslant 0$ and all n

$$|\, u_n(t)| \leqslant |\, u_0\,| + |\, S(s)\,u_0\,| \qquad \text{and} \qquad |\, \hat{u}_n(t)| \leqslant |\, \hat{u}_0\,| + |\, S(s)\,\hat{u}_0\,|.$$

Then by the dominated convergence Theorem

$$\int_0^{s+n^{-1}} |\, u_n - \hat{u}_n\,|^2\, d\sigma \to \int_0^s |\, S(\sigma)u_0 - S(\sigma)\hat{u}_0\,|^2\, d\sigma, \qquad \text{as} \quad n \to \infty.$$

Thus (25) yields, as $n \to \infty$,

$$\tfrac{1}{2}\,|\, u_0 - \hat{u}_0\,|^2 - \tfrac{1}{2}\,|\, S(s)u_0 - S(s)\hat{u}_0\,|^2 \geqslant \alpha \int_0^s |\, S(\sigma)u_0 - S(\sigma)\hat{u}_0\,|^2\, d\sigma,$$

for any u_0 and $\hat{u}_0$ in $D(A)$ and any $s > 0$. Replacing u_0 by $S(t)\,u_0$ and $\hat{u}_0$ by $S(t)\,\hat{u}_0$ and using the semigroup property we see that

$$\tfrac{1}{2} \mid S(t)u_0 - S(t)\hat{u}_0 \mid^2 - \tfrac{1}{2} \mid S(t+s)u_0 - S(t+s)\hat{u}_0 \mid^2$$
$$\geqslant \alpha \int_t^{t+s} \mid S(\sigma)u_0 - S(\sigma)\hat{u}_0 \mid^2 d\sigma,$$

for any u_0, $\hat{u}_0$ in $D(A)$ and any $s, t \geqslant 0$.

Now put

$$F(t) = \mid S(t)\,u_0 - S(t)\,\hat{u}_0 \mid^2,$$

hence

$$F(t+s) - F(t) \leqslant -2\alpha \int_t^{t+s} F(\sigma)\,d\sigma.$$

Since, according to [3, Theorem 3.1] F is absolutely continuous on any compact interval in $\mathbb{R}^+$, we deduce

$$(dF/dt)(t) \leqslant -2\alpha F(t), \qquad \text{a.e. in } \mathbb{R}^+.$$

Then $F(t) \leqslant e^{-2\alpha t}F(0)$ and (23) follows.

4. L^p-Stability

Assume $0 \in A0$ and let $p \in [1, +\infty[$; as usual, we say that Eq. (1) is L^p-stable if for any $f \in L^p(\mathbb{R}^+, H)$ and any $u_0 \in \overline{D(A)}$ the weak solution on $\mathbb{R}^+$ of the Cauchy problem

$$(du/dt) + Au \ni f, \qquad u(0) = u_0 \tag{26}$$

belongs to $L^p(\mathbb{R}^+, H)$.

By arguments similar to those used in the proof of Theorem 1 we may establish

Proposition 3. *Assume that Eq.* (1) *is L^p-stable for a $p \in [1, +\infty[$. Then, for any $u_0 \in \overline{D(A)}$ and any $f \in L^p(\mathbb{R}^+, H)$ the solution u of* (26) *belongs to $L^r(\mathbb{R}^+, H)$ for all $r \in]p, +\infty]$ and tends to* 0 *as $t \to +\infty$ (so that the trivial solution $w = 0$ of* (6) *is globally asymptotically stable in the sense of Liapunov).*

Theorem 4. *If Eq.* (1) *is L^1-stable, the operator $[u_0, f] \mapsto u$ is Lipschitz from $\overline{D(A)} \times L^1(\mathbb{R}^+, H)$ to $L^\infty(\mathbb{R}^+, H)$. If Eq.* (1) *is L^2-stable, the operator $[u_0, f] \mapsto u$ is locally Hölder of order 2^{-1} from $\overline{D(A)} \times L^2(\mathbb{R}^+, H)$ to $L^\infty(\mathbb{R}^+, H)$.*

Proof. The first assertion is obvious by [3, Lemma 3.1].

Assume that Eq. (1) is L^2-stable. For any $u_0 \in \overline{D(A)}$ denote by $\mathscr{B}_{u_0}$ the operator of $L^2(\mathbb{R}^+, H)$ defined by $f \in \mathscr{B}_{u_0} u$ if u is a weak solution of (26) on $\mathbb{R}^+$. As in the proof of Theorem 1, we may see that the operator $f \mapsto \mathscr{B}_{u_0}^{-1} f$ is locally Hölder of order 2^{-1} from $L^2(\mathbb{R}^+, H)$ to $L^\infty(\mathbb{R}^+, H)$.

Let $[u_0, f]$ and $[\hat{u}_0, \hat{f}]$ be arbitrary in $\overline{D(A)} \times L^2(\mathbb{R}^+, H)$. Denote by u, $\hat{u}$, and v the solutions assigned to $[u_0, f]$, $[\hat{u}_0, \hat{f}]$, and $[u_0, \hat{f}]$, respectively. The second assertion of Theorem 4 follows now by

$$\| u - \hat{u} \|_{L^\infty(\mathbb{R}^+, H)} \leqslant \| u - v \|_{L^\infty(\mathbb{R}^+, H)} + \| v - \hat{u} \|_{L^\infty(\mathbb{R}^+, H)},$$

since, according to [3, Lemma 3.1],

$$\| v - \hat{u} \|_{L^\infty(\mathbb{R}^+, H)} \leqslant | u_0 - \hat{u}_0 |.$$

By using Lemma 1, it is easy to see that if A is strongly monotone and $0 \in A0$, then for any $p \in [1, +\infty[$ Eq. (1) is L^p-stable and the operator $[u_0, f] \mapsto u$ from $\overline{D(A)} \times L^p(\mathbb{R}^+, H)$ to $L^p(\mathbb{R}^+, H)$ is Lipschitz. L^p-stability for $p \in [1, 2]$ is still ensured when the strong monotony condition is weakened:

THEOREM 5. *Assume that $0 \in A0$ and that there exist $\alpha > 0$, $\beta > 0$, and $\rho > 0$ such that*

$$(x, y) \geqslant \alpha \, | x |^2, \qquad \text{for all } [x, y] \in A \text{ with } | x | \leqslant \rho, \tag{27}$$

and

$$(x, y) \geqslant \beta, \qquad \text{for all } [x, y] \in A, \text{ with } | x | > 2^{-1}\rho. \tag{28}$$

Then, for any $p \in [1, 2]$, Eq. (1) is L^p-stable.

Proof. Let u_0 be arbitrary in $\overline{D(A)}$ and denote by u the weak solution of (26) on $\mathbb{R}^+$.

We claim that if $| u | > 2^{-1}\rho$ on $[\tau, \xi]$, then

$$2^{-1} \, | u(t)|^2 \leqslant 2^{-1} \, | u(\tau)|^2 - \beta(t - \tau) + \int_\tau^t | f(\sigma)| \, | u(\sigma)| \, d\sigma, \qquad \forall t \in [\tau, \xi]. \tag{29}$$

Clearly it suffices to prove (29) under the additional assumption that u is a strong solution on $[\tau, \xi]$. Now in this case we have

$$f(t) - (du/dt)(t) \in Au(t) \qquad \text{a.e. in } [\tau, \xi],$$

and so (28) implies

$$(f(t) - (du/dt)(t), u(t)) \geqslant \beta, \qquad \text{a.e. in } [\tau, \xi],$$

hence (29) follows by integration on $]\tau, t[$.

Denote by q the conjugate of p and choose $\tau > 0$ such that

$$\left\{\int_\tau^{+\infty} |f|^p \, d\sigma\right\}^{1/p} \leqslant \begin{cases} \min(\beta^{1/2}, 3^{-1}\rho), & \text{if } p = 1, \\ \min(\beta^{1/2}, 3^{-1}\rho(q\alpha)^{1/q}), & \text{if } p \in]1, 2]. \end{cases} \tag{30}$$

Since $q \geqslant 2$, we may choose $\xi > \tau$ such that

$$2^{-1} |u(\tau)|^2 + \beta^{1/2}(\xi - \tau)^{1/q} |u(\tau)| + 2^{-1}\beta(\xi - \tau)^{2/q} - \beta(\xi - \tau) < 0. \tag{31}$$

We now prove the existence of an $s \in [\tau, \xi]$ such that $|u(s)| \leqslant 2^{-1}\rho$. For if not, (29) holds. By [3, Lemma 3.1],

$$|u(t)| \leqslant |u(\tau)| + \int_\tau^t |f| \, d\sigma, \qquad \forall t \geqslant \tau,$$

hence by applying the Hölder inequality and (30) we have for all $t \geqslant \tau$

$$\begin{aligned} \int_\tau^t |f| \, |u| \, d\sigma &\leqslant |u(\tau)| \int_\tau^t |f| \, d\sigma + 2^{-1} \left(\int_\tau^t |f| \, d\sigma\right)^2 \\ &\leqslant \beta^{1/2}(t - \tau)^{1/q} |u(\tau)| + 2^{-1}\beta(t - \tau)^{2/q}. \end{aligned}$$

Then (29) combined with (31) yield $|u(\xi)| < 0$, a contradiction.

Put

$$\theta = \sup\{\sigma \geqslant s : |u(t)| < \rho, \forall t \in [s, \sigma]\}.$$

By arguments similar to those used in the proof of Theorem 2, we see that $\theta = +\infty$, hence $|u(t)| < \rho$ for all $t \geqslant s$. Then by using (27) we deduce

$$|u(t)| \leqslant e^{-\alpha(t-s)} |u(s)| + \int_s^t e^{-\alpha(t-\sigma)} |f(\sigma)| \, d\sigma, \qquad \forall t \geqslant s,$$

and now Theorem 5 follows easily.

Remark 5. It is easy to see that under the assumptions of Theorem 5, the globally asymptotic stability of the trivial solution of (6) is uniform (i.e., the solution u of

$$(du/dt) + Au \ni 0, \qquad u(0) = u_0$$

tends to 0 as $t \to +\infty$, uniformly with respect to u_0 in any bounded set of $\overline{D(A)}$).

Note also that the conditions in Theorem 5 hold when A is the subdifferential of a function φ which satisfies the conditions in Remark 3.

We now discuss L^∞-stability. We say that Eq. (1) is L^∞-stable if for any $u_0 \in \overline{D(A)}$ and any $f \in L^\infty(\mathbb{R}^+, H)$ the weak solution u of (26) belongs to $L^\infty(\mathbb{R}^+, H)$. Note that we do not require here $0 \in A0$.

If A is strongly monotone then Eq. (1) is L^∞-stable and the operator $[u_0, f] \mapsto u$ from $\overline{D(A)} \times L(\mathbb{R}^+, H)$ to $L^\infty(\mathbb{R}^+, H)$ is Lipschitz. This follows again by using Lemma 1.

PROPOSITION 4. *Assume A coercive. Then Eq. (1) is L^∞-stable and the operator which assigns to $[u_0, f] \in \overline{D(A)} \times L^\infty(\mathbb{R}^+, H)$ the weak solution $u \in L^\infty(\mathbb{R}^+, H)$ of (26) on $\mathbb{R}^+$ is bounded on any bounded set.*

Proof. Coerciveness means that there exists $x_0 \in H$ such that

$$| x |^{-1}(y, x - x_0) \to +\infty, \qquad \text{as } | x | \to +\infty \text{ with } [x, y] \in A$$

(see [3]). It follows that for any $\alpha > 0$ and any $\beta > 0$ there exists $\rho \geqslant 2^{1/2}\alpha$, such that for all $[x, y] \in A$ with $| x - x_0 | \geqslant 2^{-1/2}\rho$ we have

$$x \neq 0 \qquad \text{and} \qquad [1 + | x |^{-1} | x_0 |]\beta - | x |^{-1}(y, x - x_0) \leqslant 0. \tag{32}$$

To prove Proposition 4, it suffices to show that the weak solution u of (26) on $\mathbb{R}^+$ satisfies

$$| u(t) - x_0 | \leqslant \rho, \qquad \forall t \geqslant 0, \tag{33}$$

whenever $[u_0, f] \in \overline{D(A)} \times L^\infty(\mathbb{R}^+, H)$ verifies

$$| u_0 - x_0 | < \alpha \qquad \text{and} \qquad \| f \|_{L^\infty(\mathbb{R}^+, H)} < \beta.$$

Assume $[u_0, f]$ satisfies the above conditions and consider an arbitrary $T > 0$. Clearly there exists a sequence (f_n) of H-valued simple functions on $[0, T]$ such that $f_n \to f$ in $L^1(0, T, H)$ and $\| f_n \|_{L^\infty(\mathbb{R}^+, H)} < \beta$ for all n. There exists also a sequence (u_{0n}) of members of $D(A)$ such that $u_{0n} \to u_0$ and $| u_{0n} - x_0 | < \alpha$ for all n. Denote by u_n the solution of

$$(du_n/dt) + Au_n \ni f_n, \qquad u_n(0) = u_{0n} \text{ on } [0, T].$$

Since $u_n \to u$ uniformly on $[0, T]$, to prove (33) it suffices to show that for all n

$$| u_n(t) - x_0 | \leqslant \rho, \qquad \forall t \in [0, T]. \tag{34}$$

Note that, according to [3, Proposition 3.3], the solution u_n is a strong one and is Lipschitz on $[0, T]$.

Assume there exists n such that (34) does not hold. Put $V(t) = 2^{-1} | u_n(t) - x_0 |^2$,

$$\tau = \inf\{t \in [0, T] : V(t) > 2^{-1}\rho^2\},$$

$$\sigma = \sup\{t \in [0, \tau] : V(t) < 2^{-2}\rho^2\}.$$

The assumption that (34) does not hold combined with the continuity of V and $V(0) < 2^{-2}\rho^2$ yields

$$0 < \sigma < \tau < T, \qquad V(\sigma) = 2^{-2}\rho^2, \qquad V(\tau) = 2^{-1}\rho^2,$$

and

$$2^{-2}\rho^2 \leqslant V(t) \leqslant 2^{-1}\rho^2, \qquad \forall t \in [\sigma, \tau],$$

hence

$$| u_n(t) - x_0 | \geqslant 2^{-1/2}\rho, \qquad \forall t \in [\sigma, \tau]. \tag{35}$$

Since V is absolutely continuous, we have a.e. in $[\sigma, \tau]$

$$\begin{aligned}(dV/dt)(t) &= ((du_n/dt)(t), u_n(t) - x_0) = (f_n(t), u_n(t) - x_0)\\ &\quad - (f_n(t) - (du_n/dt)(t), u_n(t) - x_0) \leqslant | u_n(t)| \{[1 + | u_n(t)|^{-1} | x_0 |]\beta\\ &\quad - | u_n(t)|^{-1}(f_n(t) - (du_n/dt)(t), u_n(t) - x_0)\}.\end{aligned}$$

Then by (35) combined with

$$f_n(t) - (du_n/dt)(t) \in Au_n(t) \qquad \text{a.e. in } [\sigma, \tau],$$

we may apply (32) to see that $dV/dt \leqslant 0$ a.e. in $[\sigma, \tau]$. It follows that V is decreasing on $[\sigma, \tau]$, in contradiction with $V(\sigma) = 2^{-2}\rho^2$ and $V(\tau) = 2^{-1}\rho^2$. This completes the proof of Proposition 4.

Remark 6. Coercivity of A does not imply continuity of the operator $[u_0, f] \mapsto u$. For instance, in $H = \mathbb{R}$ define A by $Ax = x$, if $x \leqslant 0$, $Ax = 0$, if $x \in]0, 1[$, $A1 = \mathbb{R}^+$ and $Ax = \varnothing$, if $x > 1$. Clearly A is maximal monotone and coercive. It is easy to see that the solution u_ξ of

$$(du_\xi/dt) + Au_\xi \ni \xi, \qquad u_\xi(0) = 0, \qquad (\xi \in]0, +\infty[)$$

does not converge in $L^\infty(\mathbb{R}^+)$ to the solution u of

$$(du/dt) + Au \ni 0, \qquad u(0) = 0$$

as $\xi \to 0$.

In the sequel we assume that A is the subdifferential of a function $\varphi: H \to]-\infty, +\infty]$, which is proper, convex, and lower semicontinuous. Under this condition we will establish the converse of Proposition 4.

LEMMA 4. *Let $\xi \in H$. If there exists $u_0 \in \overline{D(A)}$ such that the solution u of*

$$(du/dt) + Au \ni \xi, \qquad u(0) = u_0$$

satisfies $\liminf_{t\to+\infty} | u(t)| < \infty$, *then* $\xi \in R(A)$.

Proof. The assumption $\liminf_{t\to+\infty} |u(t)| < \infty$ implies the existence of a sequence of positive numbers $t_n \to +\infty$ such that $(u(t_n))$ is bounded in H. Define $\varphi_\xi: H \to]-\infty, +\infty]$ by $\varphi_\xi(x) = \varphi(x) - (x, \xi)$. Clearly the function φ_ξ is proper, convex, lower-semicontinuous, and $\partial\varphi_\xi = A - \xi$, so that (36) may be written as

$$(du/dt) + \partial\varphi_\xi(u) \ni 0, \qquad u(0) = u_0. \tag{37}$$

Then by applying [3, Theorems 3.1 and 3.2], we see that u is differentiable on the right on $]0, +\infty[$,

$$(d^+u/dt)(t) + Au(t) \ni \xi, \qquad \forall t > 0, \tag{38}$$

$| d^+u/dt |$ decreases and for any $\delta > 0$ we have

$$(u(t), \xi) - \varphi(u(t)) = \int_\delta^t | d^+u/dt |^2 \, d\sigma + (u(\delta), \xi) - \varphi(u(\delta)), \qquad \forall t \geqslant \delta. \tag{39}$$

Since $-\varphi$ is majorized by an affine continuous function and $(u(t_n))$ is bounded, we infer from (39) (with t replaced by t_n) that $| d^+u/dt | \in L^2(\delta, +\infty)$. This, combined with the fact that $| d^+u/dt |$ decreases, yields $d^+u/dt \to 0$ as $t \to +\infty$. By (38)

$$u(t_n) \in A^{-1}(\xi - (d^+u/dt)(t_n))$$

and we may now apply [7, Lemma 2.3] to A^{-1} and see that $\xi \in R(A)$.

Remark 7. By applying [4] (to (37)) and Lemma 4, we deduce the asymptotical behavior of the solutions of (36): if $\xi \in R(A)$ then for any $u_0 \in D(A)$ the solution u of (36) converges weakly in H to a point of $A^{-1}\xi$ as $t \to +\infty$; if $\xi \in R(A)$ then for any $u_0 \in \overline{D(A)}$ the solution u of (36) satisfies $| u(t)| \to +\infty$ as $t \to +\infty$. It follows that surjectivity of $A = \partial\varphi$ is a necessary condition for Eq. (1) to be L^∞-stable. When $\dim H < \infty$, the above condition is also sufficient, for, according to [3, Remark 2.3], surjectivity and coercivity of $A = \partial\varphi$ are then equivalent properties.

THEOREM 6. *The following three conditions on $A = \partial\varphi$ are equivalent*:

(i) *A is coercive*;

(ii) *Eq.* (1) *is L^∞-stable and the operator which assigns to $[u_0, f] \in D(A) \times L^\infty(\mathbb{R}^+, H)$ the solution $u \in L^\infty(\mathbb{R}^+, H)$ of* (26) *is bounded on any bounded set*;

(iii) *there exists $u_0 \in \overline{D(A)}$ such that for any $\xi \in H$, the solution u of* (36) *belongs to $L^\infty(\mathbb{R}^+, H)$ and the operator $\xi \mapsto u$ is bounded on any bounded set.*

Proof. The implication (i) $\Rightarrow$ (ii) follows by Proposition 4 and (ii) $\Rightarrow$ (iii) is obvious.

According to [3, Proposition 2.14], to prove (iii) $\Rightarrow$ (i) it suffices to show that (iii) implies the existence of a single-valued operator B defined on the whole of H such that $B \subset A^{-1}$ and such that B is bounded on any bounded set. Now let ξ be arbitrary in H and denote by u the solution of (36) on $\mathbb{R}^+$. By Lemma 4, $\xi \in R(A)$ so that, as noted in Remark 7, u converges weakly to a point $B\xi \in A^{-1}\xi$ as $t \to +\infty$. It follows

$$| B\xi | \leqslant \liminf_{t\to+\infty} | u(t)| \leqslant \| u \|_{L^\infty(\mathbb{R}^+, H)} . \tag{40}$$

Clearly B is everywhere defined. Condition (iii) combined with (40) guarantees that B is bounded on any bounded set.

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